Wigner quantum oscillators

1994 J. Phys. A: Math. Gen. 27 977

(http://iopscience.iop.org/0305-4470/27/3/035)

J. Phys. A: Math. Gen. 27 (1994) 977–983. Printed in the UK

Wigner quantum oscillators

T D Palev†‡§ and N I Stoilova‡§

† International Centre for Theoretical Physics, 34100 Trieste, Italy
‡ Applied Mathematics and Computer Science, University of Ghent, B-9000 Ghent, Belgium

Received 10 September 1993

Abstract. We present three groups of non-canonical quantum oscillators. The position and momentum operators of each group generate basic Lie superalgebras, namely $sl(1/3)$, $osp(1/6)$ and $osp(3/2)$. The $sl(1/3)$ oscillators have finite energy spectrum and finite dimensions. The $osp(1/6)$ oscillators are related to the para-Bose statistictics. The internal angular momentum s of the $osp(3/2)$ oscillators takes no more than three (half)integer values. In a particular representation $s = 1/2$.

In 1950 Wigner published a paper entitled: 'Do the equations of motion determine the quantum mechanical commutation relations ?' [1]. The question to answer was whether for a (one-dimensional) quantum system with a Hamiltonian

$$H = (p^2/2m) + V(q)$$

one can derive the canonical commutation relations (CCRs)

$$[p, q] = -\mathrm{i}\hbar \tag{1}$$

assuming that the Hamiltonian equations

$$\dot{q} = p/m \qquad \dot{p} = -\partial V/\partial q \tag{2}$$

and the Heisenberg equations (in the corresponding picture)

$$\dot{q} = -(\mathrm{i}/\hbar)[q, H] \qquad \dot{p} = -(\mathrm{i}/\hbar)[p, H] \tag{3}$$

hold. The point of Wigner was that (2) and (3) have a more immediate physical significance than (1). The inverse is known to be true [2]: from (1) and (2) (respectively (1) and (3)) one derives (3) (respectively (2)). Therefore the question in fact was whether one can generalize the concept of a quantum system in a logically consistent way. Considering as an example a one-dimensional harmonic oscillator ($m = \hbar = 1$), $H = \frac{1}{2}(q^2 + p^2)$, Wigner has shown that such a generalization is indeed possible and in fact he found a family of non-canonical solutions, labelled with an arbitrary non-negative number E_0, the energy of the ground state. In terms of the operators

$$a^+ = \frac{1}{\sqrt{2}}(q - \mathrm{i}p) \qquad a^- = \frac{1}{\sqrt{2}}(q + \mathrm{i}p) \tag{4}$$

---

§ Permanent address: Institute for Nuclear Research and Nuclear Energy, Boulevard Tsarigradsko Chausse 72, 1784 Sofia, Bulgaria; E-mail palev@bgearn.bitnet

0305-4470/94/030977+07$07.50

the result of Wigner can be stated as follows. The Hamiltonian equations (2) are identical to Heisenberg equation (3) for all (representations of the) operators $a^{\pm}$, which satisfy the relations [3]

$$[\{a^{\xi}, a^{\eta}\}, a^{\epsilon}] = (\epsilon - \xi)a^{\eta} + (\epsilon - \eta)a^{\xi}. \tag{5}$$

Here and throughout $\xi, \eta, \epsilon = \pm$ or ± 1; $[x, y] = xy - yx, \{x, y\} = xy + yx$. The case $E_0 = 1/2$ corresponds to the canonical case, i.e. only for this value of E_0 are $a^{\pm}$ ordinary Bose operators, $[a^-, a^+] = 1$.

Although Wigner's paper attracted some immediate attention [4], most of the investigations following it remained in the frame of the one-dimensional case [5] (see also [3] for other references). Certainly, one can immediately generalize the above ideas to any n-dimensional oscillator and, in particular, to a three-dimensional oscillator with a Hamiltonian

$$H = \frac{(p_1)^2 + (p_2)^2 + (p_3)^2}{2m} + \frac{m\omega^2}{2}[(r_1)^2 + (r_2)^2 + (r_3)^2] \tag{6}$$

assuming simply that the coordinates and momenta corresponding to different degrees of freedom commute with each other: $[(p_i, r_i), (p_j, r_j)] = 0$ for $i \neq j$. There also exist, however, other, non-trivial generalizations. One such three-dimensional oscillator with quite unconventional properties was studied by one of us (TDP) in [6] (see also below). In the present paper we shall give an example of another non-canonical three-dimensional Wigner oscillator, which has an interesting physical property: the spin of the oscillator is 1/2. The oscillators considered in [6, 7] and the one we are going to study here are particular cases of what we call Wigner quantum oscillators (and, more generally, the Wigner quantum system). The oscillator is said to be a Wigner quantum oscillator if the following conditions are fulfilled.

(i) The state space W is a Hilbert space. The physical observables are Hermitian operators in W.

(ii) The Hamiltonian equations and the Heisenberg equations are identical (as operator equations) in W.

(iii) The internal angular momentum (the spin) of the oscillator $\boldsymbol{s} = (s_1, s_2, s_3)$ is a linear function of the position operators $\boldsymbol{r} = (r_1, r_2, r_3)$ and the momentum $\boldsymbol{p} = (p_1, p_2, p_3)$, so that $\boldsymbol{s}$, $\boldsymbol{r}$ and $\boldsymbol{p}$ transform as vectors: $[s_j, c_k] = i\sum_{l=1}^{3} \varepsilon_{jkl} c_l$, $c_k = s_k, r_k, p_k, i, j, k = 1, 2, 3$.

(iv) The spectrum of H is bounded from below.

The underlying mathematical structure of the oscillators which we consider is one and the same. It is related to the representation theory of some basic Lie superalgebras [8]. As we shall see, this is also the case for the canonical three-dimensional oscillator. In order to outline the link with the Lie superalgebras (see also [6]), we introduce in place of the unknown $\boldsymbol{p}$ and $\boldsymbol{r}$ new unknown operators

$$a_k^{\pm} = \sqrt{\frac{m\omega}{2\hbar}} r_k \mp \frac{\mathrm{i}}{\sqrt{2m\omega\hbar}} p_k \qquad k = 1, 2, 3. \tag{7}$$

In terms of $a_k^{\pm}$, which we call creation and annihilation operators (CAOs), the Hamiltonian (6) reads:

$$H = \frac{1}{2}\omega\hbar \sum_{k=1}^{3} \{a_k^+, a_k^-\}. \tag{8}$$

Condition (ii) yields ($n = 3$; $k = 1, 2, 3$):

$$\sum_{i=1}^{n}[\{a_i^+, a_i^-\}, a_k^\pm] = \pm 2a_k^\pm. \tag{9}$$

Equations (9) are a unique consequence from the Hamiltonian equations

$$\dot{\boldsymbol{p}} = -m\omega^2 \boldsymbol{r} \qquad \dot{\boldsymbol{r}} = \boldsymbol{p}/m \tag{10}$$

and the Heisenberg equations

$$\dot{\boldsymbol{p}} = -(\mathrm{i}/\hbar)[\boldsymbol{p}, H] \qquad \dot{\boldsymbol{r}} = -(\mathrm{i}/\hbar)[\boldsymbol{r}, H] \tag{11}$$

independently of the properties of the unknown CAOs $a_k^\pm$. They are equal time relations, the time dependence being $a_k^\varepsilon(t) = \mathrm{e}^{\mathrm{i}\varepsilon\omega t}a_k^\varepsilon(0)$, $\varepsilon = \pm$. Hence equations (9) hold, if they are fulfilled at, say, $t = 0$.

In order to be slightly more general, let us denote by $F(n)$ the associative algebra with unity, generators $a_1^\pm, \ldots, a_n^\pm$ and relations (9). Then any representation of $F(3)$ is a candidate for a Wigner oscillator or, more precisely, the CAOs of any Wigner oscillator give a representation of $F(3)$. In such a case the representation space of $F(3)$ (the corresponding $F(3)$ module) is a state space of the oscillator. For definiteness we call the algebra $F(n)$ an (n-dimensional) free oscillator algebra. Thus, as a first step, one has to find the representations of $F(3)$ and then select those of them for which conditions (i), (ii) and (iv) also hold. It turns out this is not an easy problem and, in fact, it is so far unsolved. Here we list three classes of solutions.

1. Class 1 solutions: *osp*(1/6) oscillators

Let $F_1(n)$ be the (free unital) associative superalgebra with odd generators $a_1^\pm, \ldots, a_n^\pm$ and relations

$$\begin{aligned}&[\{a_i^\xi, a_j^\eta\}, a_k^\epsilon] = \delta_{ik}(\epsilon - \xi)a_j^\eta + \delta_{jk}(\epsilon - \eta)a_i^\xi \qquad i, j, k = 1, \ldots, n\\ &\xi, \eta, \epsilon = \pm \text{ or } \pm 1.\end{aligned} \tag{12}$$

The operators (12) satisfy equations (9) and therefore $F_1(n)$ is a factor algebra of $F(n)$. Consequently any representation of $F_1(n)$ is a representation of $F(n)$. Observe that Wigner's solutions belong to this class ($n = 1$). The canonical solution, namely the one in which the CAOs are Bose operators is also from this class. It is easily verified that operators (12) are para-Bose (pB) operators [9]. Their main algebraic property stems from the observation that the subspace

$$B_1 = \text{lin.env.}\{a_i^\xi, \{a_j^\eta, a_k^\varepsilon\} | i, j, k = 1, \ldots, n\xi, \eta, \epsilon = \pm\} \subset F_1(n) \tag{13}$$

is a Lie superalgebra [10] with odd generators as the pB operators. This algebra is isomorphic to one of the basic Lie superalgebras in the classification of Kac [8], namely to the orthosymplectic *LS* $osp(1/2n) \equiv B(0/n)$, whereas $F_1(n)$ is its universal enveloping algebra [11]. As a result the representation theory of any n pairs of pB operators is completely equivalent to the representation theory of the *LS* $osp(1/2n)$. It is another question that for physical reasons one has to select a subclass of representations, which in the case $n = 3$ should satisfy the conditions (i)–(iv). Unfortunately not much is known about the representations of $osp(1/6)$ and, more generally, about $osp(1/2n)$ (apart from the full classification of the finite-dimensional modules [8]). The only technique to construct new representations from the Fock representation was developed by Green [9] through the Green ansatz [12]. It leads, however, to reducible representations and is realized in tensor products of Fock spaces. The representation with statistics of order p corresponds to the irreducible representation of $osp(1/2n)$, containing the highest weight vector (which is the vacuum) in the tensor product of p copies of Fock spaces (considered as $osp(1/2n)$ modules). There exists, however, no effective methods for extracting this representation from the reducible tensor product representation. This may be the reason why pB oscillators with dimensions higher than one have not so far been considered. The important conclusion for us is that there exist solutions of the free oscillator algebra $F(n)$ with operators, which generate the basic Lie superalgebras $osp(1/2n)$, namely an LS from the class B in the Cartan–Kac classification [8]. This naturally leads to the idea of trying to find solutions of equations (9) with representations of other LSs from the same class B or from the other classes of basic Lie superalgebras.

2. Class 2 solutions: $sl(1/3)$ oscillators [6]

Let $F_2(n)$ be the associative superalgebra with generators $a_1^\pm, \ldots, a_n^\pm$ and relations

$$\begin{aligned} &[\{a_i^+, a_j^-\}, a_k^+] = \delta_{jk} a_i^+ - \delta_{ij} a_k^+ \\ &[\{a_i^+, a_j^-\}, a_k^-] = -\delta_{ik} a_j^- + \delta_{ij} a_k^- \\ &\{a_i^+, a_j^+\} = \{a_i^-, a_j^-\} = 0. \end{aligned} \tag{14}$$

These operators also satisfy equations (9) and therefore $F_2(n)$ is another factor algebra of the free oscillator algebra $F(n)$. The subspace

$$A = \text{lin.env.}\{a_i^\pm, \{a_j^+, a_k^-\} | i, j, k = 1, \ldots, n\} \subset F_2(n) \tag{15}$$

is a Lie superalgebra with odd generators $a_1^\pm, \ldots, a_n^\pm$, which is isomorphic to the Lie superalgebra $sl(1/n) \equiv A(0, n-1)$ from the class A of the basic Lie superalgebras. $F_2(n)$ is its universal enveloping algebra. Hence any representation of $sl(1/n)$ gives a solution of equations (9). The condition (i) restricts the class of representations to the finite-dimensional representations of $sl(1/3)$, which are explicitly known [13]. The internal angular momentum (condition (iii)) is $s_i = -i \sum_{k,l=1}^3 \varepsilon_{ikl} \{a_k^+, a_l^-\}$. A class of state spaces, labelled with any non-negative integer p, was studied in [6]. The corresponding oscillator, one can call it the $sl(1/3)$ oscillator, is very unconventional. We mention some of its properties. The spectrum of the Hamiltonian is finite; it has no more than four different eigenvalues. The square distance operator $r^2 = (r_1)^2 + (r_2)^2 + (r_3)^2$ is an integral of motion. Its maximal eigenvalue is $(r_{\max})^2 = 3p\hbar/2m\omega$. Therefore the oscillator is confined in the space. It resembles in this respect a wavelet (see [14] and the references therein). The spin of the oscillator is either 0 or 1. Finally, the coordinates r_1, r_2, r_3 do not commute with each other, so that the position of the oscillating particle cannot be localized. The particle is smeared with a certain probability along a sphere with a fixed radius.

3. Class 3 solutions: *osp*(3/2) oscillators

Another new class of solutions of the compatibility equations (9), i.e. of condition (ii), is given with the set of all possible representations of operators $a_1^\pm, a_1^\pm, a_3^\pm$, which satisfy the following relations ($\epsilon = \pm$ or $\pm 1, i, j, k = 1, 2, 3$):

$$[\{a_i^+, a_j^-\}, a_k^\epsilon] = \tfrac{2}{3}\delta_{ik}a_j^\epsilon - \tfrac{2}{3}\delta_{jk}a_i^\epsilon + \tfrac{2}{3}\delta_{ij}\epsilon a_k^\epsilon$$

$$[\{a_i^\epsilon, a_i^\epsilon\}, a_k^{-\epsilon}] = -\tfrac{4}{3}\epsilon a_k^\epsilon$$

$$[\{a_i^\epsilon, a_i^\epsilon\}, a_k^\epsilon] = 0$$

$$\{a_i^\epsilon, a_j^\epsilon\} = 0 \qquad i \neq j \tag{16}$$

$$\{a_i^+, a_j^-\} = -\{a_j^+, a_i^-\} \qquad i \neq j$$

$$\{a_1^+, a_1^-\} = \{a_2^+, a_2^-\} = \{a_3^+, a_3^-\}$$

$$(a_1^\epsilon)^2 = (a_2^\epsilon)^2 = (a_3^\epsilon)^2. \tag{16}$$

We denote by $F_3(3)$ the infinite-dimensional associative superalgebra with generators $a_1^\pm, a_1^\pm, a_3^\pm$ and relations (16). The grading on $F_3(3)$ is induced from the requirement that the CAOs are odd generators. Consider the subspace

$$B_3 = \text{lin.env.}\{a_i^\xi, \{a_j^\eta, a_k^\epsilon\} | i, j, k = 1, \dots, n \xi, \eta, \epsilon = \pm\} \subset F_3(3) \tag{17}$$

and turn it into a Lie superalgebra with the supercommutator which is natural for any associative superalgebra, namely $\langle a, b\rangle = ab - (-1)^{\alpha\beta} ba$, where $\alpha = \deg(a)$, $\beta = \deg(b)$. Elsewhere we shall show that B_3 is isomorphic to the orthosymplectic Lie superalgebra $osp(3/2)$ and that $F_3(3)$ is its universal enveloping algebra. Therefore we call this oscillator an $osp(3/2)$ oscillator. The angular momentum satisfying condition (iii) reads as

$$s_j = -\frac{3i}{4}\sum_{k,l=1}^{3} \epsilon_{jkl}\{a_j^-, a_k^+\} \qquad j = 1, 2, 3. \tag{18}$$

The $osp(3/2)$ modules (= representation spaces) for which the conditions (i) and (iv) also hold are infinite-dimensional. They are labelled with all possible pairs (p, q), where p is an arbitrary non-negative half-integer, and q is any negative real number, such that $p + 2q \leqslant 0$. All representation spaces $W(p, q)$ (among others) have been described in [15]. The energy of the oscillator depends only on the value of q and is

$$E_n = \omega\hbar(n - 2q) \qquad n = 0, 1, 2, \dots. \tag{19}$$

Depending on the representation, the ground energy can be arbitrarily close to zero (for $p = 0$ and very small negative q), but never zero. The spin s depends mainly on p and takes at most three different values. More precisely, the spin content within each state space $W(p, q)$ reads:

(a) $p = 0$ $\qquad s = 0, 1$

(b) $p = 1/2, p + 2q = 0$ $\qquad s = 1/2$

(c) $p = 1/2, p + 2q < 0$ $\qquad s = 1/2, 3/2$

(d) $p = 1, p + 2q = 0$ $\qquad s = p - 1, p$

(e) $p \geqslant 1, p + 2q < 0$ $\qquad s = p - 1, p, p + 1.$

The derivation of the above results, together with the multiplicities of the states will be given elsewhere. Here we consider explicitly the most simple and, maybe, the most interesting representation, the one corresponding to the spin 1/2 (case (b)). This representation is sometimes referred to as a metaplectic representation [16]. An orthonormed basis in this state space $W(1/2, -1/4) \equiv W(1/2)$ is given with the set of all vectors $|n, s_3\rangle$, where $n = 0, 1, 2, \ldots$ labels the energy of the state and $s_3 = \pm\frac{1}{2}$ is the value of the third projection of the spin.

The transformations of the basis states under the action of the CAOs reads:

$$
\begin{aligned}
a_1^-|n,s_3\rangle &= \frac{2}{\sqrt{3}}(-1)^n s_3\sqrt{n}|n-1,-s_3\rangle & a_1^+|n,s_3\rangle &= \frac{2}{\sqrt{3}}(-1)^n s_3\sqrt{n+1}|n+1,-s_3\rangle \\
a_2^-|n,s_3\rangle &= \frac{\mathrm{i}}{\sqrt{3}}(-1)^n\sqrt{n}|n-1,-s_3\rangle & a_2^+|n,s_3\rangle &= \frac{\mathrm{i}}{\sqrt{3}}(-1)^n\sqrt{n+1}|n+1,-s_3\rangle \\
a_3^-|n,s_3\rangle &= \frac{1}{\sqrt{3}}\sqrt{n}|n-1,s_3\rangle & a_3^+|n,s_3\rangle &= \frac{1}{\sqrt{3}}\sqrt{n+1}|n+1,s_3\rangle.
\end{aligned}
\tag{20}
$$

From (8) and (20) one derives $H|n, s_3\rangle = \omega\hbar(n+1/2)|n, s_3\rangle$. Thus, the energy spectrum of the oscillator in this particular representation is the same as for a one-dimensional harmonic oscillator:

$$E_n = \omega\hbar(n + 1/2) \qquad n = 0, 1, 2, \ldots. \tag{21}$$

The eigensubspace $W_n(1/2)$ of the Hamiltonian with energy E_n is spanned on $|n, s_3\rangle, s_3 = \pm 1/2$ and it is closed under the action of the spin operators. It carries a two-dimensional irreducible representation of the spin $su(2)$ algebra with generators s_1, s_2, s_3. The state space $W(1/2)$ is an infinite direct sum of spin 1/2 modules,

$$W(1/2) = \oplus_{n=0}^{\infty} W_n(1/2). \tag{22}$$

Clearly this particular $osp(3/2)$ oscillator is very different from the canonical three-dimensional oscillator. The next table demonstrates this. By $W_n, n = 0, 1, 2, \ldots$ we denote the eigensubspace of the Hamiltonian with energy E_n.

	Canonical oscillator	$Osp(3/2)$ oscillator	
Energy	$E_n = \omega\hbar(n + 3/2)$	$E_n = \omega\hbar(n + 1/2)$	(23)
Spin content of W_n	$s = n, n-2, n-4, \ldots, 1$ (or 0)	$s = 1/2$	

The purpose of the present paper was to show, using simple examples, that Wigner's ideas to study more general quantum systems, the Wigner quantum systems in our terminology, are very rich in their origin. If, for instance, one considers a non-canonical two-particle system with internal variables, which have the properties of an $sl(1/3)$ oscillator [6], then the two-particle system has finite space dimensions, it behaves like a system of two (non-relativistic) quarks, confined in the space. The $osp(3/2)$ oscillator, viewed in the same way, gives a model of a spin-1/2 system, which has a classical analogue: two non-canonical point particles curl around each other and the resulting angular momentum of the composite system is 1/2. Hence this is a model of a spin (among several others; see [17] and the references therein).

One may think that the freedom in constructing such more general quantum systems is very large. As far as the three-dimensional oscillator is concerned, we may say that the oscillators considered here exhaust all Wigner oscillators, for which the position and the momentum operators generate simple Lie superalgebras [18]. If one goes beyond the harmonic potentials, it is an open question whether those interactions for which (non-canonical) Wigner quantum systems exists.

Elsewhere we shall show that Wigner's ideas can be extended to any number of particles. In particular for oscillator-like interactions between the constituents one finds solutions for which the composite system has finite dimensions and therefore behaves very much like a nucleus.

Acknowledgments

We are grateful to Professor Vanden Berghe for the possibility of working at the Department of Applied Mathematics and Computer Science, University of Ghent. One of us (TDP) would like to thank Professor Abdus Salam, the International Atomic Energy Agency and Unesco for the hospitality at the International Centre for Theoretical Physics. It is a pleasure to thank Dr J Van der Jeugt for stimulating discussions.

This work was supported by the European Community, contract no. ERB-CIPA-CT92-2011 (Cooperation in Science and Technology with Central and Eastern European Countries) and Grant Φ-215 of the Bulgarian Foundation for Scientific Research.

References

[1] Wigner E P 1950 *Phys. Rev.* **77** 711
[2] Ehrenfest P 1927 *Z. Phys.* **4** 455
[3] Ohnuki Y and Kamefuchi S 1982 *Quantum Field Theory and Papastatistics* (Berlin: Springer; Tokyo: University of Tokyo Press)
[4] Yang L M 1951 *Phys. Rev.* **84** 788
[5] Ohnuki Y and Kamefuchi S 1978 *J. Math. Phys.* **19** 67; 1979 *Z. Phys.* C **2** 367.
Okubo S 1980 *Phys. Rev.* D **22** 919
Mukunda N, Sudarshan E C G, Sharma J K and Mehta C L 1980 *J. Math. Phys.* **21** 2386
Ohnuki Y and Watanabe S 1992 *J. Math. Phys.* **33** 3653
[6] Palev T D 1982 *J. Math. Phys.* **23** 1778; 1982 *Czech. J. Phys.* B **23** 680
[7] Kamupingene A H, Palev T D and Tsaneva S P 1986 *J. Math. Phys.* **27** 2067
[8] Kac V G 1978 *Lecture Notes in Mathematics* **626** 597
[9] Green H S 1953 *Phys. Rev.* **90** 270
[10] Omote M, Ohnuki Y and Kamefuchi S 1948 *Prog. Theor. Phys.* **56** 1948
[11] Ganchev A Ch and Palev T D 1980 *J. Math. Phys.* **21** 797
[12] Greenberg O W and Messiah A M L 1965 *Phys. Rev.* B **138** 1155
[13] Palev T D 1985 *J. Math. Phys.* **26** 1640; 1986 *J. Math. Phys.* **27** 1994; 1987 *J. Math. Phys.* **28** 272; 1987 *J. Math. Phys.* **28** 2280; 1988 *J. Math. Phys.* **29** 2589
Palev T D 1987 *Funct. Anal. Appl.* **211** 245 (English translation)
[14] Barut A O 1991 *Symposium on the Foundations of Modern Physics* ed P Lahti and P Mitelstaedt (Singapore: World Scientific) p 31
Barut A O and Bracken A J 1992 *Found. Phys.* **22** 1267
Barut A O 1992 *Phys. Lett.* **17A** 1
[15] Ky N A, Palev T D and Stoilova N I 1992 *J. Math. Phys.* **33** 1841
[16] Van der Jeugt J 1984 *J. Math. Phys.* **25** 3334
[17] Barut A O 1993 *Preprint* ICTP IC/93/104
[18] Palev T D and Stoilova N I 1993 *Classical and Quantum Systems-Foundations and Symmetries* ed H D Doebner, W Scherer and Fr Schroeck Jr (Singapore: World Scientific) p 318